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Journal of Approximation Theory 136 (2005) 159–181
Journal of Approximation Theory
www.elsevier.com/locate/jat

On best restricted range approximation in continuous complex-valued function spaces

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Received 19 August 2004; accepted 7 July 2005

Communicated by Frank Deutsch

Abstract

To provide a Kolmogorov-type condition for characterizing a best approximation in a continuous complex-valued function space, it is usually assumed that the family of closed convex sets in the complex plane used to restrict the range satisfies a strong interior-point condition, and this excludes the interesting case when some Ω_t is a line-segment or a singleton. The main aim of the present paper is to remove this restriction by virtue of a study of the notion of the strong CHIP for an infinite system of closed convex sets in a continuous complex-valued function space.

1. Introduction

Throughout this paper $C(Q)$ will denote the Banach space of all complex-valued continuous functions on a compact metric space Q endowed with the uniform norm (the "Sup-norm"). Let $\mathcal{P}$ denote a finite-dimensional subspace of $C(Q)$, and let $\{\Omega_t : t \in Q\}$ be a family of nonempty closed convex sets in the complex plane $\mathbb{C}$. Set

$$\mathcal{P}_\Omega = \{p \in \mathcal{P} : p(t) \in \Omega_t \text{ for each } t \in Q\}. \tag{1.1}$$

The captioned problem is that of finding an element $p^* \in \mathcal{P}_\Omega$ for a function $f \in C(Q)$

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doi:10.1016/j.jat.2005.07.002

such that

$$\|f - p^*\| = \inf_{p \in \mathcal{P}_\Omega} \|f - p\| \tag{1.2}$$

(such a p^* is called a best restricted range approximation to f from $\mathcal{P}$ with respect to $\{\Omega_t\}$). This problem was first presented and formulated by Smirnov and Smirnov in [24,25]; their approach followed the standard path for the corresponding issue in the real-valued continuous function space theory. In [24], while it was pointed out that this problem for the general class of restrictions was quite difficult, they took up the special case when each Ω_t is a disc in $\mathbb{C}$. Later, in a series of papers by them and by others [26–28,11,14], a more general class of $\{\Omega_t\}$ has been considered but each of them is still under a general strong interior-point condition assumption that there exists an element $\bar{p}$ of $\mathcal{P}_\Omega$ such that $\operatorname{int} \cap_{t\in Q} (\Omega_t - \bar{p}(t)) \neq \emptyset$ (hence $\operatorname{int} \Omega_t \neq \emptyset$ for each $t \in Q$). This unfortunately excludes the interesting case when some Ω_t is a line-segment or a singleton in $\mathbb{C}$. Our results in Section 3 further relax the restriction by allowing the interesting case just mentioned. Letting

$$C_t = \{u \in C(Q) : u(t) \in \Omega_t\} \quad \text{for each } t \in Q, \tag{1.3}$$

we note that $\{\mathcal{P}, C_t : t \in Q\}$ is a family of closed convex sets in $C(Q)$ with the intersection $\mathcal{P} \bigcap (\bigcap_{t\in Q} C_t) = \mathcal{P}_\Omega$. The main aim of this paper is to provide some characterizations for p^* to satisfy (1.2) in a reasonable case (under appropriate continuity assumption of the set-valued mapping $t \mapsto \Omega_t$, and a suitably relaxed interior-point condition). One such characterization is given, as in the corresponding real case, by a condition of the Kolmogorov type. Our results are obtained here by virtue of a study of the strong CHIP (the strong conical hull intersection property) for an infinite family of closed convex sets in a Banach space. The notion of the strong CHIP was first introduced by Deutsch et al. [7,8] for a finite family of closed convex sets in a Euclidean space (or a Hilbert space) and was recently extended by Li and Ng in [14] to an arbitrary family of closed convex sets in a Banach space. In [16], this notion was studied extensively and some useful sufficient conditions for the strong CHIP were established.

We end this introduction with a short remark that having obtained the characterization results as presented in Section 3, the issue of the uniqueness of solutions for the corresponding problems can be addressed along a well-established path (cf. [11]) and we need not further elaborate here.

2. Notations and preliminary results

We begin with the notations used in this paper, most of which are standard (cf. [5,10]). In particular, we assume that X is a complex (or real at times) Banach space. For a set Z in X (or in $\mathbb{R}^n$), the interior (*resp.* relative interior, closure, convex hull, convex cone hull, linear hull, affine hull, boundary, relative boundary) of Z is denoted by $\operatorname{int} Z$ (*resp.* $\operatorname{ri} Z$, $\overline{Z}$, $\operatorname{conv} Z$, $\operatorname{cone} Z$, $\operatorname{span} Z$, $\operatorname{aff} Z$, $\operatorname{bd} Z$, rb Z); the normal cone of Z at z_0 is denoted by $N_Z(z_0)$ and defined by

$$N_Z(z_0) = \{x^* \in X^* : \operatorname{Re} \langle x^*, z - z_0 \rangle \leqslant 0 \quad \text{for each } z \in Z\}. \tag{2.1}$$

The distance from z_0 to Z is denoted by $dZ(z_0)$.

Our main tools are the following Theorems 2.1 and 2.2 taken from [16, Corollaries 4.2 and 5.1]. It would be convenient for us to repeat some of the definitions introduced in [16] as well as some other more standard notions in this regard. Let I denote an index-set which is assumed to be a compact metric space. A family $\{C, C_i : i \in I\}$ is called a closed convex set system with base-set C (CCS-system with base-set C) if C and C_i are nonempty closed convex subsets of X for each $i \in I$.

Definition 2.1. A CCS-system $\{C, C_i : i \in I\}$ (with base-set C) is said to satisfy:

(i) the interior-point condition if

$$C \bigcap \left(\bigcap_{i \in I} \operatorname{int} C_i \right) \neq \emptyset; \tag{2.2}$$

(ii) the strong interior-point condition if

$$C \bigcap \left(\operatorname{int} \bigcap_{i \in I} C_i \right) \neq \emptyset; \tag{2.3}$$

(iii) the weak–strong interior-point condition with the pair (I_1, I_2) if there exist two disjoint finite subsets I_1 and I_2 of I such that each C_i $(i \in I_2)$ is a polyhedron and

$$\operatorname{ri} C \bigcap \left(\operatorname{int} \bigcap_{i \in I \setminus (I_1 \cup I_2)} C_i \right) \bigcap \left(\bigcap_{i \in I_1} \operatorname{ri} C_i \right) \bigcap_{i \in I_2} C_i \neq \emptyset. \tag{2.4}$$

Any point $\bar{x}$ belonging to the set on the left-hand side of (2.2) (resp. (2.3), (2.4)) is called an interior point (resp. a strong interior point, a weak–strong interior point with the pair (I_1, I_2)) of the CCS-system $\{C, C_i : i \in I\}$.

It is trivial that (2.2) $\Longrightarrow$ (2.3). The converse also holds in some cases, one of which will be described in terms of the continuity of some set-valued functions (cf. [16]). For set-valued functions there are many different notions of continuity. In Definitions 2.2 and 2.3 below, we recall two frequently used ones. We assume that Q is a compact metric space.

Definition 2.2. Let $F : Q \to 2^X$ be a set-valued function defined on Q and let $t_0 \in Q$. Then F is said to be

(i) lower semicontinuous at t_0, if, for any $y_0 \in F(t_0)$ and any $\varepsilon > 0$, there exists an open neighbourhood $U(t_0)$ of t_0 such that, for each $t \in U(t_0)$, $\mathbf{B}(y_0, \varepsilon) \cap F(t) \neq \emptyset$.

(ii) upper semicontinuous at t_0 if, for any open neighbourhood V of $F(t_0)$, there exists an open neighbourhood $U(t_0)$ of t_0 such that $F(t) \subseteq V$ for each $t \in U(t_0)$.

(iii) lower (resp. upper) semicontinuous on Q if it is lower (resp. upper) semicontinuous at each $t \in Q$.

Definition 2.3 (*cf. Singer [23, p. 55]*). Let $F : Q \to 2^X$ be a set-valued function defined on Q and let $t_0 \in Q$. Then F is said to be

(i) upper Kuratowski semicontinuous at t_0 if, for any sequence $\{t_k\} \subseteq Q$, the relations $\lim_{k\to\infty} t_k = t_0$, $\lim_{k\to\infty} x_{t_k} = x_{t_0}$, $x_{t_k} \in F(t_k)$, $k = 1, 2, \ldots$ imply $x_{t_0} \in F(t_0)$.

(ii) lower Kuratowski semicontinuous at t_0 if, for any sequence $\{t_k\} \subseteq Q$, the relations $\lim_{k\to\infty} t_k = t_0$, $y_0 \in F(t_0)$ imply $\lim_{k\to\infty} d_{F}(t_k)(y_0) = 0$;

(iii) Kuratowski continuous at t_0 if F is both upper Kuratowski semicontinuous and lower Kuratowski semicontinuous at t_0.

(iv) Kuratowski continuous on Q if it is Kuratowski continuous at each point of Q.

Remark 2.1. Clearly,

(i) F is upper semicontinuous $\Longrightarrow F$ is upper Kuratowski semicontinuous.
(ii) F is lower semicontinuous $\Longleftrightarrow F$ is lower Kuratowski semicontinuous.

Moreover, the converse of (i) holds provided that the union $\cup_{t\in Q} F(t)$ is compact.

Let $\{A_i : i \in J\}$ be a family of subsets of X. The set $\sum_{i\in J} A_i$ is defined by

$$\sum_{i\in J} A_i = \begin{cases} \left\{\sum_{i\in J_0} a_i : \quad a_i \in A_i, \quad J_0 \subseteq J \text{ being finite}\right\} & \text{if } J \neq \emptyset, \\ \{0\} & \text{if } J = \emptyset. \end{cases} \tag{2.5}$$

Definition 2.4. Let $\{C_i : i \in I\}$ be a collection of convex subsets of X and $x \in \bigcap_{i\in I} C_i$. The collection is said to have

(a) the strong CHIP at x if

$$N_{\bigcap_{i\in I} C_i}(x) = \sum_{i\in I} N_{C_i}(x). \tag{2.6}$$

(b) the strong CHIP if it has the strong CHIP at each point of $\cap_{i\in I} C_i$.

Theorem 2.1. *Let $x_0 \in C \cap \left(\cap_{i\in I} C_i\right)$. The system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0 if the following conditions are satisfied*:

(a) *The system $\{C, C_i : i \in I\}$ satisfies the weak–strong interior-point condition with (I_1, I_2).*
(b) *The set-valued mapping $i \mapsto C_i$ is lower semicontinuous on I.*
(c) *At least one of the sets in the family $\{C, C_i : i \in I_1\}$ is finite-dimensional.*

Theorem 2.2. *Suppose that the CCS-system $\{C, C_i : i \in I\}$ satisfies the interior-point condition*, $\dim C < +\infty$ *and that the set-valued function $i \mapsto (aff\, C) \cap C_i$ is Kuratowski continuous*. *Then the system $\{C, C_i : i \in I\}$ has the strong CHIP*.

We end this section with two results on characterizations of the strong CHIP of a (possibly infinite) system $\{C, C_i : i \in I\}$ of closed convex sets. The first result, which is valid in a general Banach space and will be used in the next section, is given in terms of the optimality conditions of a constrained best approximation while the second result in the Hilbert space setting is given as a dual formulation of a constrained best approximation (see for example, [3,4,7–9,12–15,17,18]). To this end, we need a well-known result on the characterization of the best approximation by a convex set in X, which was established independently by

Deutsch [6] and Rubenstein [20] (see also [1]). For a closed convex subset W of X, let P_W denote the projection operator defined by

$$P_W(x) = \{y \in W : \|x - y\| = d_W(x)\}.$$

Where $d_W(x)$ denotes the distance from x to W. Recall that the duality map J from X to 2^{X^*} is defined by

$$J(x) := \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \ \|x^*\| = \|x\|\}. \tag{2.7}$$

Proposition 2.1. *Let W be a closed convex set in X. Then for any $x \in X$, $z_0 \in P_W(x)$ if and only if $z_0 \in W$ and there exists $x^* \in J(x - z_0)$ such that* $\mathrm{Re}\,\langle x^*, z - z_0\rangle \leqslant 0$ *for any $z \in W$, that is, $J(x - z_0) \cap N_W(z_0) \neq \emptyset$. In particular, when X is smooth, $z_0 \in P_W(x)$ if and only if $z_0 \in W$ and $J(x - z_0) \in N_W(z_0)$.*

Theorem 2.3. *Let $K = C \cap (\cap_{i\in I} C_i)$ and $x_0 \in K$. Consider the following statements.*

(i) *The system $\{C, C_i : i \in I\}$ has the strong CHIP at x_0.*
(ii) *For each $x \in X$, $x_0 \in P_K(x)$ if and only if*

$$J(x - x_0) \bigcap \left(N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0)\right) \neq \emptyset. \tag{2.8}$$

(iii) *For each $x \in X$, $x_0 \in P_K(x)$ if and only if*

$$J(x - x_0)|_{C-x_0} \bigcap \left(N_C(x_0)|_{C-x_0} + \sum_{i\in I} N_{C_i}(x_0)|_{C-x_0}\right) \neq \emptyset. \tag{2.9}$$

Then the following implications hold.

(1) (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$(iii).
(2) (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$(iii) *if X is both reflexive and smooth.*

Proof. Note the following equivalence:

$$\begin{array}{ll} & J(x - x_0) \bigcap \left(N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0)\right) \neq \emptyset \\ \Longleftrightarrow & J(x - x_0)|_{C-x_0} \bigcap \left(N_C(x_0)|_{C-x_0} + \sum_{i\in I} N_{C_i}(x_0)|_{C-x_0}\right) \neq \emptyset. \end{array} \tag{2.10}$$

Indeed, implication $\Longrightarrow$ in (2.10) is trivial; hence it suffices to show the converse implication. Thus, let $x^* \in J(x - x_0)$ be such that $x^*|_{C-x_0} \in J(x - x_0)|_{C-x_0} \bigcap \left(N_C(x_0)|_{C-x_0} + \sum_{i\in I} N_{C_i}(x_0)|_{C-x_0}\right)$. Then there exist $x_0^* \in N_C(x_0)$, a finite subset J of I and $x_i^* \in N_{C_i}(x_0)$ for each $i \in J$ such that $x^*|_{C-x_0} = \sum_{i=0}^m x_i^*|_{C-x_0}$. Write $y^* = x^* - \sum_{i=0}^m x_i^*$. Then $y^* \in N_C(x_0)$ and so $x^* = y^* + \sum_{i=0}^m x_i^* \in N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0)$. Hence, $x^* \in J(x - x_0) \bigcap \left(N_C(x_0) + \sum_{i\in I} N_{C_i}(x_0)\right)$. Therefore (2.10) is true.

Now, using (2.10), one can complete the proof in the same way as that given for [15, Theorem 3.1]. □

For the remainder of this section, let X denote a Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$), and we consider $X^* = X$ as usual. In particular, the normal cone of a nonempty set Z at z_0 can be redefined as $N_Z(z_0) = \{y \in X : \ \mathrm{Re}\,\langle y, z - z_0\rangle \leqslant 0 \quad \text{for all } z \in Z\}$. Let $I(x_0) = \{i \in I : \ x_0 \in \mathrm{bd}\, C_i\}$. Then, similar to the proof of [14, Theorem 4.1], we obtain the following theorem.

Theorem 2.4. *Let X be a Hilbert space, $K = C \cap (\cap_{i \in I} C_i)$ and let $x_0 \in K$. Then the following statements are equivalent.*

(i) *The system $\{C, C_i : \ i \in I\}$ has the strong CHIP at x_0.*
(ii) *For any $x \in X$, $P_K(x) = x_0$ if and only if there exists a finite (possibly empty) set $I_0 \subseteq I$ such that $P_C(x - \sum_{i \in I_0} h_i) = x_0$ for some $h_i \in N_{C_i}(x_0)$ for each $i \in I_0$.*

Now, let C be a closed convex set in X, $\{h_i : \ i \in I\} \subset X \setminus \{0\}$ and let $\{\Omega_i : \ i \in I\}$ be a family of nonempty closed convex subsets of the scalar field. Define

$$\widehat{C}_i = \{x \in X : \ \langle x, h_i\rangle \in \Omega_i\}, \quad i \in I, \tag{2.11}$$

and

$$\widehat{K} = C \bigcap \left(\bigcap_{i \in I} \widehat{C}_i \right). \tag{2.12}$$

Let $x_0 \in \widehat{K}$. For convenience, we shall write $\widetilde{h}_i(\cdot)$ for the function $\langle h_i, \cdot\rangle$ on X, and h_i^0 for the scalar $\langle h_i, x_0\rangle$. Then we have the following assertion:

$$N_{\widehat{C}_i}(x_0) = \{\overline{\alpha} h_i : \ \alpha \in N_{\Omega_i}(h_i^0)\} \quad \text{for each } i \in I. \tag{2.13}$$

This assertion was proved in the proof of [14, Theorem 4.2]. Here we give a direct and much simpler proof. In fact, it is direct that the set on the left-hand side contains the one on the right-hand side of (2.13). To show the converse inclusion, let $h_i^{\perp}$ denote the orthogonal complement of h_i and let $x^* \in N_{\widehat{C}_i}(x_0)$. Then, for each $x \in h_i^{\perp}$ and $\gamma \in \mathbb{C}$, $\mathrm{Re}\,\langle x^*, \gamma x\rangle \leqslant 0$ since $\gamma x + x_0 \in \widehat{C}_i$; hence $x^* \perp h_i^{\perp}$ and $x^* = \bar{\alpha} h_i$ for some scalar $\alpha \in \mathbb{C}$. Since, for each $\beta \in \Omega_i$, there exists $x \in \widehat{C}_i$ such that $\langle h_i, x\rangle = \beta$, we have that

$$\mathrm{Re}\,\overline{\alpha}(\beta - h_i^0) = \langle x^*, x - x_0\rangle \leqslant 0.$$

This means that $\alpha \in N_{\Omega_i}(h_i^0)$. Therefore x^* belongs to the set on the right-hand side of (2.13) and (2.13) is proved. Thus, by (2.13) and Theorem 2.4, we immediately obtain the following perturbation theorem, which was given in [14]. Note that the proof here is much simpler than that in [14].

Corollary 2.1. *Let X be a Hilbert space and let $x_0 \in \widehat{K}$, where $\widehat{K}$ is defined by* (2.12). *Then the following statements are equivalent.*

(i) *The collection of closed convex sets $\{C, \widehat{C}_i : \ i \in I\}$ has the strong CHIP at x_0. and*
(ii) *For any $x \in X$, $P_{\widehat{K}}(x) = x_0$ if and only if there exists a finite (possibly empty) set $I_0 \subseteq I$ such that $P_C(x - \sum_{i \in I_0} \overline{\alpha}_i h_i) = x_0$ for some $\alpha_i \in N_{\Omega_i}(h_i^0)$ for each $i \in I_0$.*

3. Characterization for constrained approximation in complex-valued function spaces

Let $C(Q)$ denote the Banach space of all complex-valued continuous functions on a compact metric space Q endowed with the uniform norm:

$$\|f\| = \max_{t\in Q} |f(t)| \quad \text{for each } f \in C(Q). \tag{3.1}$$

Let $\mathcal{P}$ be an n-dimensional subspace of $C(Q)$ and $\{\Omega_t : t \in Q\}$ a family of nonempty closed convex sets in the complex plane $\mathbb{C}$. For brevity, we write $\{\Omega_t\}$ for $\{\Omega_t : t \in Q\}$. Note that, for each $t \in Q$, Ω_t is either a point or a linear-segment, or a "planar" convex set (of real dimension 2) in the complex plane $\mathbb{C}$. Set

$$\mathcal{P}_\Omega = \{p \in \mathcal{P} : p(t) \in \Omega_t \quad \text{for each } t \in Q\}. \tag{3.2}$$

The problem considered here is that of finding an element $p^* \in \mathcal{P}_\Omega$ for a function $f \in C(Q)$ such that

$$\|f - p^*\| = \inf_{p\in\mathcal{P}_\Omega} \|f - p\|, \tag{3.3}$$

(such a p^* is called a best-restricted range approximation to f from $\mathcal{P}$ with respect to $\{\Omega_t\}$; see [24,28,11,14]).

We assume that

$$Q = Q_S \bigcup Q_E \bigcup Q_N, \tag{3.4}$$

where

$$Q_S = \{t \in Q : \Omega_t \text{ is a singleton}\},$$

$$Q_E = \{t \in Q \setminus Q_S : \operatorname{int}\Omega_t = \emptyset\},$$

$$Q_N = \{t \in Q : \operatorname{int}\Omega_t \neq \emptyset\}.$$

We also assume for the whole section that

$$Q_S \cup Q_E \text{ is finite.} \tag{3.5}$$

We introduce some short notation of conditions for easy reference.

- IC_0: $\mathcal{P}$ contains the constant functions and there exists an element $\bar{p} \in \mathcal{P}_\Omega$ such that $\bar{p}(t) \in \operatorname{int}\Omega_t$ for each $t \in Q$, that is,

$$0 \in \bigcap_{t\in Q} \operatorname{int}(\Omega_t - \bar{p}(t)). \tag{3.6}$$

- IC: There exists an element $\bar{p} \in \mathcal{P}_\Omega$ such that

$$0 \in \operatorname{int}\left(\bigcap_{t\in Q_N} (\Omega_t - \bar{p}(t))\right) \bigcap \left(\bigcap_{t\in Q_E} \operatorname{ri}(\Omega_t - \bar{p}(t))\right). \tag{3.7}$$

- UKC: The set-valued function $t \mapsto \Omega_t$ is upper Kuratowski semicontinuous on Q.
- LKC: The set-valued function $t \mapsto \Omega_t$ is lower Kuratowski semicontinuous on Q.
- KC: The set-valued function $t \mapsto \Omega_t$ is Kuratowski continuous on Q.

We will see later that these conditions closely relate to some corresponding properties of the CCS-system $\{\mathcal{P}, C_t : t \in Q\}$ in $C(Q)$, where C_t is defined by (1.3). Let $f \in C(Q)$ and $p^* \in \mathcal{P}_\Omega$. We fix this pair of functions f, p^* in what follows. Define

$$\sigma(t) = f(t) - p^*(t) \quad \text{for each } t \in Q. \tag{3.8}$$

Set

$$M(\sigma) = \{t \in Q : |\sigma(t)| = \|\sigma\|\}$$

and

$$B(p^*) = \{t \in Q : p^*(t) \in \mathrm{bd}\Omega_t\}, \quad B^{rb}(p^*) = \{t \in Q \setminus Q_S : p^*(t) \in \mathrm{rb}\Omega_t\}.$$

(Here we adopt the convention that $\mathrm{bd}\,\Omega_t = \Omega_t$ if Ω_t is a singleton.) Note that

$$B^{rb}(p^*) = (B(p^*) \cap Q_N) \cup \{t \in Q_E : p^*(t) \in \mathrm{rb}\Omega_t\} \tag{3.9}$$

and in particular that $B^{rb}(p^*) \subseteq B(p^*)$. Moreover, $B^{rb}(p^*) = B(p^*)$ in the case when Q_S and Q_E are empty (e.g., when IC_0 holds).

Let $\mathrm{span}_R\,(\Omega_t - p^*(t))$ denote the real subspace spanned by $\Omega_t - p^*(t)$ in $\mathbb{C}$. Then $\mathrm{span}_R\,(\Omega_t - p^*(t))$ is the whole complex plane $\mathbb{C}$ if $t \in Q_N$, a line in $\mathbb{C}$ if $t \in Q_E$ and a singleton $\{0\}$ if $t \in Q_S$. Set

$$\mathcal{P}_R = \{p \in \mathcal{P} : p(t) \in \mathrm{span}_R\left(\Omega_t - p^*(t)\right) \text{ for each } t \in Q_E \cup Q_S\}. \tag{3.10}$$

Note that $\mathcal{P}_R$ is a real subspace of $\mathcal{P}$ and that $\mathcal{P}_R = \mathcal{P}$ if $Q = Q_N$. Let m denote the real dimension of $\mathcal{P}_R$: $\dim_R \mathcal{P}_R = m$, and let $\psi_1, \ldots, \psi_m$ be a real basis of $\mathcal{P}_R$, that is, each element of $\mathcal{P}_R$ can be uniquely expressed as a real linear combination of $\psi_1, \ldots, \psi_m$. Moreover, let $\{\phi_1, \ldots, \phi_n\}$ be a (complex) basis of $\mathcal{P}$, that is, each element of $\mathcal{P}$ can be uniquely expressed as a complex linear combination of $\phi_1, \ldots, \phi_n$.

We define

$$\tau(t) = \{\tau \in -N_{\Omega_t}(p^*(t)) : |\tau| = 1\} \quad \text{for each} \quad t \in Q. \tag{3.11}$$

Note that if $t \in Q_N \cap B(p^*)$ and $\tau \in \tau(t)$ then

$$\mathrm{Re}\,\overline{\tau}(z - p^*(t)) > 0 \tag{3.12}$$

for all $z \in \mathrm{int}\,\Omega_t$. Since $\mathrm{int}\,\Omega_t = \emptyset$ if $t \in Q \setminus Q_N$, we have to define two more set-valued functions from Q to the unit sphere of $\mathbb{C}$:

$$\tau_r(t) = \begin{cases} \tau(t) & \text{for each } t \in Q \setminus Q_E, \\ \{\tau \in \mathbb{C} : |\tau| = 1,\ \mathrm{Re}\,\overline{\tau}(z - p^*(t)) > 0 & \\ \quad \forall z \in \mathrm{ri}\,\Omega_t\} & \text{for each } t \in Q_E \end{cases} \tag{3.13}$$

and

$$\tau_r^+(t) = \begin{cases} \tau(t) & \text{for each } t \in Q \setminus Q_E, \\ \emptyset & \text{for each } t \in Q_E \text{ with } p^*(t) \in \operatorname{ri}\Omega_t, \\ \frac{z-p^*(t)}{|z-p^*(t)|} & \text{for each } t \in Q_E \text{ with } p^*(t) \in \operatorname{rb}\Omega_t,\ z \in \Omega_t \setminus p^*(t). \end{cases} \tag{3.14}$$

(Note that $\frac{z-p^*(t)}{|z-p^*(t)|}$ does not depend on the particular choice of z as Ω_t is a line-segment for $t \in Q_E$.)

Remark 3.1. (i) For any $t \in Q$, $\tau(t) \neq \emptyset \iff t \in B(p^*)$.

(ii) For any $t \in Q_E$,

$$\tau_r(t) \neq \emptyset \iff t \in B^{rb}(p^*) \iff \tau_r^+(t) \text{ is a singleton.} \tag{3.15}$$

(iii) If $t \in B^{rb}(p^*) \cap Q_E$ and $\tau \in -N_{\Omega_t}(p^*(t))$ with $|\tau| = 1$, then

$$\tau \notin \tau_r(t) \iff \operatorname{Re} \overline{\tau}(z - p^*(t)) = 0 \quad \text{for each } z \in \Omega_t \iff \operatorname{Re} \overline{\tau}(z - p^*(t)) = 0 \\ \text{for some } z \in \Omega_t. \tag{3.16}$$

(iv) For any $t \in Q$, $\tau_r^+(t)$ is compact

$$\tau_r^+(t) \subseteq \tau_r(t) \subseteq \tau(t). \tag{3.17}$$

Let $t \in B^{rb}(p^*) \cap Q_E$, $\tau \in \tau_r(t)$ and let $\operatorname{Pr}_t(\tau)$ denote the projection of τ on the subspace $\operatorname{span}_R(\Omega_t - p^*(t))$. Then $\operatorname{Pr}_t(\tau) \neq 0$,

$$\frac{\operatorname{Pr}_t(\tau)}{|\operatorname{Pr}_t(\tau)|} \in \tau_r^+(t) \quad \text{and} \quad \operatorname{Re} z\overline{\tau} = \operatorname{Re} z\overline{\operatorname{Pr}_t(\tau)} \\ \text{for each } z \in \operatorname{span}_R\left(\Omega_t - p^*(t)\right). \tag{3.18}$$

For each $t \in Q$, let $\mathbf{c}(t) \subset \mathbb{C}^n$, $\mathbf{c}_r(t) \subset \mathbb{R}^m$ and $\mathbf{c}_r^+(t)$ be defined, respectively, by

$$\mathbf{c}(t) := \{(\phi_1(t)\overline{\tau}, \ldots, \phi_n(t)\overline{\tau}) : \tau \in \tau(t)\}, \tag{3.19}$$

$$\mathbf{c}_r(t) := \{(\operatorname{Re}\psi_1(t)\overline{\tau}, \ldots, \operatorname{Re}\psi_m(t)\overline{\tau}) : \tau \in \tau_r(t)\} \tag{3.20}$$

and

$$\mathbf{c}_r^+(t) := \{(\operatorname{Re}\psi_1(t)\overline{\tau}, \ldots, \operatorname{Re}\psi_m(t)\overline{\tau}) : \tau \in \tau_r^+(t)\}. \tag{3.21}$$

Set

$$\mathcal{U} = \bigcup_{t \in B(p^*)} \mathbf{c}(t), \quad \mathcal{U}_r = \bigcup_{t \in B^{rb}(p^*)} \mathbf{c}_r(t), \quad \mathcal{U}_r^+ = \bigcup_{t \in B^{rb}(p^*)} \mathbf{c}_r^+(t). \tag{3.22}$$

Note that these sets are bounded and that, by (3.17) and (3.18),

$$\mathcal{U}_r^+ \subseteq \mathcal{U}_r \subseteq \bigcup_{0<\eta\leqslant 1} \left(\eta\,\mathcal{U}_r^+\right). \tag{3.23}$$

Recalling (3.8), we define $\mathbf{b}(t) \in \mathbb{C}^n$ and $\mathbf{b}_r(t) \in \mathbb{R}^m$, respectively, by

$$\mathbf{b}(t) = (\phi_1(t), \ldots, \phi_n(t))\overline{\sigma(t)} = (\phi_1(t)\overline{\sigma(t)}, \ldots, \phi_n(t)\overline{\sigma(t)}) \quad \text{for each } t \in Q \tag{3.24}$$

and

$$\mathbf{b}_r(t) = \mathrm{Re}\,(\psi_1(t), \ldots, \psi_m(t))\overline{\sigma(t)} \quad \text{for each } t \in Q. \tag{3.25}$$

We define

$$\mathcal{V} = \{\mathbf{b}(t) : t \in M(\sigma)\}, \quad \mathcal{V}_r = \{\mathbf{b}_r(t) : t \in M(\sigma)\}. \tag{3.26}$$

Clearly they are compact sets. Set

$$\mathcal{W} = \mathcal{V} \bigcup \mathcal{U}, \quad \mathcal{W}_r = \mathcal{V}_r \bigcup \mathcal{U}_r, \quad \mathcal{W}_r^+ = \mathcal{V}_r \bigcup \mathcal{U}_r^+. \tag{3.27}$$

Note that they are bounded sets. Also, by (3.23),

$$\mathcal{W}_r^+ \subseteq \mathcal{W}_r \subseteq \bigcup_{0<t\leqslant 1} \left(t\,\mathcal{W}_r^+\right). \tag{3.28}$$

This implies that

$$\mathrm{co}\,\mathcal{W}_r^+ \subseteq \mathrm{co}\,\mathcal{W}_r \subseteq \mathrm{co} \bigcup_{0<t\leqslant 1} \left(t\,\mathcal{W}_r^+\right) \subseteq \bigcup_{0<t\leqslant 1} \left(t\,\mathrm{co}\,\mathcal{W}_r^+\right), \tag{3.29}$$

where the last inclusion can be verified by a routine verification.

Let

$$C_t = \{u \in C(Q) : u(t) \in \Omega_t\} \quad \text{for each } t \in Q. \tag{3.30}$$

Then

$$\mathcal{P}_\Omega = \mathcal{P} \bigcap \left(\bigcap_{t\in Q} C_t \right). \tag{3.31}$$

Clearly $\{\mathcal{P}, C_t : t \in Q\}$ is a CCS-system with a base-set $\mathcal{P}$. To prepare for our main result, we first give a few lemmas. These lemmas will show in particular that the conditions introduced at the beginning of this section for the system $\{\Omega_t\}$ are naturally linked to some desirable properties of the system $\{\mathcal{P}, C_t : t \in Q\}$ so that the results in Section 2 are applicable. The first of the lemmas describes the connections of the conditions IC_0, IC for the system $\{\Omega_t\}$ and the interior-point conditions for the system $\{\mathcal{P}, C_t : t \in Q\}$ while the second describes the connection of the normal cones of Ω_t and that of the corresponding C_t.

Lemma 3.1. (i) *The system* $\{\Omega_t\}$ *satisfies* IC_0 *if and only if the CCS-system* $\{\mathcal{P}, C_t : t \in Q\}$ *satisfies the interior-point condition. Furthermore,* $0 \notin \mathrm{conv}\,\mathcal{U}$ *if the system* $\{\Omega_t\}$ *satisfies* IC_0.

(ii) *The system* $\{\Omega_t\}$ *satisfies* IC *if and only if the CCS-system* $\{\mathcal{P}, C_t : t \in Q\}$ *satisfies the weak–strong interior-point condition with the pair* (Q_E, Q_S). *Furthermore,* $0 \notin \operatorname{conv}\mathcal{U}_r$ *if the system* $\{\Omega_t\}$ *satisfies* IC.

Proof. Let $\alpha > 0$ and $f_0 \in C_t$. We claim that

$$\mathbf{B}(f_0, \alpha) \subseteq C_t \Longleftrightarrow \mathbf{B}(f_0(t), \alpha) \subseteq \Omega_t \quad \text{for each } t \in Q_N, \tag{3.32}$$

$$\mathbf{B}(f_0, \alpha) \bigcap \operatorname{aff} C_t \subseteq C_t \Longleftrightarrow \mathbf{B}(f_0(t), \alpha) \bigcap \operatorname{aff} \Omega_t \subseteq \Omega_t \quad \text{for each } t \in Q_E. \tag{3.33}$$

We shall only prove (3.33) (the proof of (3.32) is similar). To do this, we need only establish the necessity part. Note first the following obvious fact:

$$\operatorname{aff} C_t = \{u \in C(Q) : u(t) \in \operatorname{aff} \Omega_t\} \quad \text{for each } t \in Q. \tag{3.34}$$

Let $t \in Q_E$ and assume that

$$\mathbf{B}(f_0, \alpha) \bigcap \operatorname{aff} C_t \subseteq C_t. \tag{3.35}$$

Let $z \in \mathbf{B}(f_0(t), \alpha) \bigcap \operatorname{aff} \Omega_t$. We have to show that $z \in \Omega_t$. By the Tietze Extension Theorem, there exists $s \in C(Q)$ such that $\|s\| = s(t) = 1$. Define

$$f(w) = f_0(w) + s(w)(z - f_0(t)) \quad \forall w \in Q.$$

Then $\|f - f_0\| \leqslant |z - f_0(t)| \leqslant \alpha$. Since $f(t) = z \in \operatorname{aff} \Omega_t$, $f \in \operatorname{aff} C_t$ by (3.34). Consequently, $f \in C_t$ and hence $z = f(t) \in \Omega_t$, as required. Therefore, our claim stands.

By (3.32), we have that

$$\operatorname{int} C_t = \{u \in C(Q) : u(t) \in \operatorname{int} \Omega_t\} \quad \text{for each } t \in Q. \tag{3.36}$$

Thus the first part of (i) is clear. Again by (3.32),

$$\operatorname{int} \bigcap_{t \in Q_N} C_t = \left\{ u \in C(Q) : u(t) \in \operatorname{int} \bigcap_{t \in Q_N} \Omega_t \right\}, \tag{3.37}$$

while, by (3.33),

$$\operatorname{ri} C_t = \{u \in C(Q) : u(t) \in \operatorname{ri} \Omega_t\} \quad \text{for each } t \in Q_E. \tag{3.38}$$

Combining (3.37) and (3.38), the first part of (ii) is also clear.

Thus, to complete the proof, it remains to show that (a): $0 \notin \operatorname{conv}\mathcal{U}_r$ if IC is satisfied and that (b): $0 \notin \operatorname{conv}\mathcal{U}$ if IC_0 is satisfied. We shall only prove (a) as the proof for (b) is similar. Suppose that there exist $\lambda_1, \ldots, \lambda_s \in [0, 1]$ with $\sum_{j=1}^s \lambda_j = 1$ and $t'_1, \ldots, t'_s \in B^{rb}(p^*)$, $\tau'_j \in \tau_r(t'_j)$, $j = 1, \ldots, s$ such that

$$\operatorname{Re} \sum_{j=1}^{s} p(t'_j)\lambda_j \overline{\tau'_j} = 0 \tag{3.39}$$

holds for each $p \in \{\psi_1, \dots, \psi_m\}$ and hence for each $p \in \mathcal{P}_R$. Assuming IC with some $\bar{p} \in \mathcal{P}_\Omega$ satisfying (3.7), let $p := \bar{p} - p^*$. Since, each $t'_j \in B^{rb}(p^*)$ and each $\tau'_j \in \tau_r(t'_j)$, we obtain, by (3.12), (3.7) and (3.13) that

$$\operatorname{Re} p(t'_j)\overline{\tau}'_j = \operatorname{Re}\left(\bar{p}(t'_j) - p^*(t'_j)\right)\overline{\tau}'_j > 0 \quad \text{for each } j = 1, \dots, s. \tag{3.40}$$

This contradicts (3.39) and hence $0 \notin \operatorname{conv}\mathcal{U}_r$. □

Lemma 3.2. *Let $t \in Q$ and assume that $p^* \in C_t$. Then*

$$N_{C_t}(p^*) = \{\overline{\alpha}e_t : \alpha \in N_{\Omega_t}(p^*(t))\}, \tag{3.41}$$

where e_t denotes the point-functional on $C(Q)$ defined by

$$\langle e_t, u\rangle = u(t) \quad \textit{for each } u \in C(Q). \tag{3.42}$$

Proof. Let $u \in C(Q)$. Let $z \in \Omega_t$ be such that $d_{\Omega_t}(u(t)) = |z - u(t)|$. By the Tietze Extension Theorem, there exists a function $w \in C(Q)$ such that $\|w\| = |u(t) - z|$ and $w(t) = u(t) - z$ (so $u - w \in C_t$). Then $d_{C_t}(u) \leqslant \|u - (u - w)\| = |z - u(t)| = d_{\Omega_t}(u(t))$. Consequently,

$$d_{C_t}(u) = d_{\Omega_t}(u(t)) \quad \text{for each } u \in C(Q) \tag{3.43}$$

as it is straightforward to verify that $d_{C_t}(u) \geqslant d_{\Omega_t}(u(t))$. Since $p^* \in C_t$ (and so $p^*(t) \in \Omega_t$), (3.43) and the proof of [14, Lemma 5.2 (iii)] imply that

$$\partial d_{C_t}(p^*) = \{\overline{\alpha}e_t \in C(Q)^* : \alpha \in \partial d_{\Omega_t}(p^*(t))\}. \tag{3.44}$$

Recalling from [5] that

$$\begin{aligned} &\partial d_{C_t}(p^*) = \{x^* \in N_{C_t}(p^*) : \|x^*\| \leqslant 1\} \quad \text{and} \\ &\partial d_{\Omega_t}(p^*(t)) = \{\alpha \in N_{\Omega_t}(p^*(t)) : |\alpha| \leqslant 1\}, \end{aligned} \tag{3.45}$$

it follows that (3.41) holds. □

Lemma 3.3. (i) *If UKC is satisfied, then the set-valued function $t \mapsto C_t$ is upper Kuratowski semicontinuous on Q.*

(ii) *If LKC is satisfied, then the set-valued function $t \mapsto C_t$ is lower Kuratowski semicontinuous on Q* (and so is the set-valued function $t \mapsto \mathcal{P} \cap C_t$ if $1 \in \mathcal{P}$).

Proof. Let $t_0 \in Q$ and $\{t_k\} \subseteq Q$ be a sequence convergent to t_0.

(i) Let $f_k \in C_{t_k}$ for each k be such that $\|f_k - \bar{f}\| \to 0$. Then, $f_k(t_k) \in \Omega_{t_k}$ for each k and $f_k(t_k) \to \bar{f}(t_0)$ as $k \to \infty$. By the condition UKC, it follows that $\bar{f}(t_0) \in \Omega_{t_0}$ and so $\bar{f} \in C_{t_0}$. This proves (i).

(ii) Let $f_0 \in C_{t_0}$ (or $f_0 \in \mathcal{P} \cap C_{t_0}$ if $1 \in \mathcal{P}$). Then $f_0(t_0) \in \Omega_{t_0}$ and, by the condition LKC, there exists $z_k \in \Omega_{t_k}$ for each k such that $|z_k - f_0(t_0)| \to 0$. Define $f_k \in C(Q)$ by

$$f_k(t) = f_0(t) + z_k - f_0(t_k) \quad \text{for each } t \in Q.$$

Thus $f_k(t_k) = z_k \in \Omega_{t_k}$ and hence $f_k \in C_{t_k}$ (and $f_k \in \mathcal{P} \cap C_{t_k}$ if $1 \in \mathcal{P}$). Moreover, we also have that

$$\|f_k - f_0\| = |z_k - f_0(t_k)| \leqslant |z_k - f_0(t_0)| + |f_0(t_0) - f_0(t_k)| \to 0.$$

Thus (ii) is proved. □

Lemma 3.4. *Suppose that the condition LKC is satisfied. Then $B(p^*)$ is closed and $\mathcal{W}$ is compact in $\mathbb{C}^n$.*

Proof. Let $\{t_k\} \subseteq B(p^*)$ and $\{\tau_k\} \subseteq \cup_{t \in B(p^*)} \tau(t)$ be such that $\tau_k \in \tau(t_k)$, $t_k \to t_0 \in Q$ and $\tau_k \to \tau \in \mathbb{C}$. Then $|\tau_k| = |\tau| = 1$. Moreover, since $Q \setminus Q_N$ is finite, we assume, without loss of generality, that each $t_k \in Q_N$. Then, for each k,

$$\operatorname{Re} \overline{-\tau_k}(z - p^*(t_k)) \leqslant 0 \quad \text{for each } z \in \Omega_{t_k}. \tag{3.46}$$

By the condition LKC, for each $z \in \Omega_{t_0}$, there exists $z_k \in \Omega_{t_k}$ such that $z_k \to z$. Noting that $p^*(t_k) \to p^*(t_0)$, it follows from (3.46) that

$$\operatorname{Re} \overline{-\tau}(z - p^*(t_0)) \leqslant 0 \ \text{ for all } z \in \Omega_{t_0}. \tag{3.47}$$

Since $p^*(t_0) \in \Omega_{t_0}$ as $p^* \in \mathcal{P}_\Omega$, this means that $-\tau \in N_{\Omega_{t_0}}(p^*(t_0))$. Since $\tau \neq 0$, this implies that $p^*(t_0) \in \operatorname{bd}\Omega_{t_0}$ and so $t_0 \in B(p^*(t_0))$. Hence, $B(p^*)$ is closed and hence $\tau \in \cup_{t \in B(p^*)} \tau(t)$. This shows that $\cup_{t \in B(p^*)} \tau(t)$ is closed and hence compact since it is bounded. By definition, it is now easily verified that $\mathcal{U}$ is compact. Since $\mathcal{V}$ is compact, it follows that $\mathcal{W}$ is compact. □

Lemma 3.5. *Suppose that the conditions LKC and IC hold. Then $B^{rb}(p^*)$ is closed and the closure of $\mathcal{W}_r^+$ is contained in $\mathcal{W}_r$.*

Proof. As in the proof of Lemma 3.4, let $\{t_k\} \subseteq B^{rb}(p^*)$ and $\tau_k \in \tau_r^+(t_k)$ for each k such that $t_k \to t_0 \in Q$ and $\tau_k \to \tau \in \mathbb{C}$. Thus, by (3.9) and (3.17), one has $\{t_k\} \subseteq B(p^*)$ and $\tau_k \in \tau(t_k)$ for each k. By Lemma 3.4, it follows that $t_0 \in B(p^*)$ and $-\tau \in N_{\Omega_{t_0}}(p^*(t_0))$ thanks to LKC. It suffices to show that $t_0 \in B^{rb}(p^*)$ and $\tau \in \tau_r(t_0)$. If $t_0 \in Q_N$, they are done by the proof of Lemma 3.4 because one then has $t_0 \in B(p^*) \cap Q_N \subseteq B^{rb}(p^*)$ and $\tau \in \tau(t_0) = \tau_r(t_0)$. Thus, we may assume henceforth that $t_0 \notin Q_N$. Note that if $t_k \in Q_E$ for infinitely many k, then, since Q_E is finite, one has $t_k = t_0$ for these k (say for all k by considering a subsequence if necessary). Hence $t_0 \in B^{rb}(p^*)$ and $\tau_k \in \tau_r(t_0)$. However, in view of (3.15), $\tau_r(t_0)$ must be a singleton in the present case, so $\tau \in \tau_r(t_0)$. Therefore, without loss of generality, we may assume that $t_k \in Q_N$ for each k. In view of (3.27), to complete the proof, it is sufficient to show that $t_0 \in Q_E$, $p^*(t_0) \in \operatorname{rb}\Omega_{t_0}$ and

$$\operatorname{Re} \overline{\tau}(z - p^*(t_0)) > 0 \quad \text{for some } z \in \operatorname{ri}\Omega_{t_0}. \tag{3.48}$$

By IC, there exists $\bar{p} \in \mathcal{P}_\Omega$ satisfying (3.7). Let $\delta > 0$ be such that

$$\mathbf{B}(0, \delta) \subset \bigcap_{t \in Q_N} (\Omega_t - \bar{p}(t)). \tag{3.49}$$

We will show that there exists an integer $N > 0$ such that

$$\mathbf{B}\left(\bar{p}(t_0) - p^*(t_0), \frac{\delta}{2}\right) \subset \bigcap_{k \geqslant N} \left(\Omega_{t_k} - p^*(t_k)\right). \tag{3.50}$$

Indeed, take $N > 0$ such that $|(\bar{p}(t_k) - p^*(t_k)) - (\bar{p}(t_0) - p^*(t_0))| < \frac{\delta}{2}$ for each $k \geqslant N$. Then

$$\mathbf{B}\left(\bar{p}(t_0) - p^*(t_0), \frac{\delta}{2}\right) \subset \mathbf{B}(\bar{p}(t_k) - p^*(t_k), \delta) \quad \text{for each } k \geqslant N. \tag{3.51}$$

On the other hand, by (3.49),

$$\mathbf{B}(\bar{p}(t_k) - p^*(t_k), \delta) \subset \Omega_{t_k} - p^*(t_k) \quad \text{for each } k. \tag{3.52}$$

Consequently, (3.50) follows from (3.51) and (3.52). Set $\Omega^* := \bigcap_{k \geqslant N} \left(\Omega_{t_k} - p^*(t_k)\right)$. Then $0 \in \operatorname{bd} \Omega^*$ and $\bar{p}(t_0) - p^*(t_0) \in \operatorname{int} \Omega^*$ by (3.50). In particular,

$$\operatorname{Re} \overline{\alpha}(\bar{p}(t_0) - p^*(t_0)) < 0 \quad \text{for each } \alpha \in N_{\Omega^*}(0) \setminus \{0\}.$$

Hence, there exists a positive number b such that, for each $\alpha \in N_{\Omega^*}(0)$ with $|\alpha| = 1$,

$$\operatorname{Re} \overline{\alpha}(\bar{p}(t_0) - p^*(t_0)) \leqslant -b < 0. \tag{3.53}$$

Since $-\tau_k \in N_{\Omega_{t_k}}(p^*(t_k))$, $|-\tau_k| = 1$ and $N_{\Omega_{t_k}}(p^*(t_k)) \subseteq N_{\Omega^*}(0)$ for each $n \geqslant N$, we have that

$$\operatorname{Re} \overline{-\tau_k}(\bar{p}(t_0) - p^*(t_0)) \leqslant -b < 0 \quad \text{for each } k \geqslant N. \tag{3.54}$$

Noting that $\tau_k \to \tau$, it follows that

$$\operatorname{Re} \overline{-\tau}(\bar{p}(t_0) - p^*(t_0)) \leqslant -b < 0. \tag{3.55}$$

Thus Ω_{t_0} contains more than one point ($\bar{p}(t_0)$, $p^*(t_0)$ being distinct members of Ω_{t_0}). It follows that Ω_{t_0} is a line-segment (recalling that $t_0 \notin Q_N$), i.e., $t_0 \in Q_E$. Consequently, by (3.7), $\bar{p}(t_0) \in \operatorname{ri} \Omega_{t_0}$. Therefore (3.48) holds by (3.55). Since $0 \neq -\tau \in N_{\Omega_{t_0}}(p^*(t_0))$ (noting $\bar{p}(t_0) \in \Omega_{t_0}$), it follows from (3.55) that $p^*(t_0)$ must be an end point of Ω_{t_0}, i.e., $p^*(t_0) \in \operatorname{rb} \Omega_{t_0}$. The proof of Lemma 3.5 is complete. □

Lemma 3.6. *Let Φ be a complex linear functional on $\mathcal{P}$ such that*

$$\operatorname{Re} \Phi(p) = 0 \quad \textit{for each } p \in \mathcal{P}_R. \tag{3.56}$$

Then there exist a nonnegative integer s with $s \leqslant 2n - m$, $\{t''_j\}_{j=1}^s \subseteq Q_E \cup Q_S$ and $\{\tau''_j\}_{j=1}^s \subset \mathbb{C} \setminus \{0\}$ with each $\tau''_j \in -N_{\Omega_{t''_j}}(p^*(t''_j))$ such that

$$\Phi(p) + \sum_{j=1}^{s} p(t''_j)\overline{\tau''_j} = 0 \quad \textit{for each } p \in \mathcal{P}. \tag{3.57}$$

Proof. We assume that $Q_E \cup Q_S \neq \emptyset$ (the result is trivial otherwise). For each $t \in Q_E$, $\text{span}_R(\Omega_t - p^*(t))$ is a line passing through the origin. Hence there exists $\tau_t \in \mathbb{C}$ with $|\tau_t| = 1$ which is "perpendicular" to $\text{span}_R(\Omega_t - p^*(t))$ in the sense that

$$\text{Re}\,\overline{\tau_t}\,\alpha = 0 \Longleftrightarrow \alpha \in \text{span}_R(\Omega_t - p^*(t)). \tag{3.58}$$

Thus, defining the real linear functional ξ_t on $\mathcal{P}$ by

$$\xi_t(p) = \text{Re}\,\overline{\tau_t}\,p(t) \quad \text{for each } p \in \mathcal{P}, \tag{3.59}$$

we can characterize the kernel of ξ_t for $t \in Q_E$:

$$p \in \text{Ker}\,\xi_t \Longleftrightarrow p(t) \in \text{span}_R(\Omega_t - p^*(t)). \tag{3.60}$$

For each $t \in Q_S$, two linear functionals on $\mathcal{P}$ (respectively, denoted by ξ_t and ξ'_t) will be useful for us. They are defined by

$$\xi_t(p) = \text{Re}\,p(t) \quad \text{for each } p \in \mathcal{P},$$

$$\xi'_t(p) = \text{Re}\,\mathrm{i}p(t) \quad \text{for each } p \in \mathcal{P},$$

where $\mathrm{i} = \sqrt{-1}$. Thus, for $t \in Q_S$,

$$p(t) = 0 \Longleftrightarrow p \in \text{Ker}\,\xi_t \bigcap \text{Ker}\,\xi'_t.$$

By (3.10), we have that

$$\mathcal{P}_R = \left(\bigcap_{t\in Q_E\cup Q_S} \text{Ker}\,\xi_t\right) \bigcap \left(\bigcap_{t\in Q_S} \text{Ker}\,\xi'_t\right). \tag{3.61}$$

It will be convenient to list the functionals

$$\{\xi_t : t \in Q_E \cup Q_S\} \bigcup \{\xi'_t : t \in Q_S\} := \{\xi^1, \xi^2, \ldots, \xi^r\}, \tag{3.62}$$

where $r = |Q_E| + 2|Q_S|$, and for example $|Q_E|$ stands for the number of elements in Q_E. Letting $q := 2n - m$, the difference of real dimensions of $\mathcal{P}$ and $\mathcal{P}_R$, one has $q \leqslant r$. Recalling that $\{\psi_1, \ldots, \psi_m\}$ is a basis of $\mathcal{P}_R$, there exist $\psi_{m+1}, \ldots, \psi_{2n} \in \mathcal{P}$ such that $\{\psi_1, \ldots, \psi_{2n}\}$ is a real basis of $\mathcal{P}$. Since $\mathcal{P}_R \cap \text{span}_R\{\psi_{m+1}, \ldots, \psi_{m+q}\} = \{0\}$, it is easy to verify that the vectors $\{\vec{a}_i : i = m+1, \ldots, m+q\} \subset \mathbb{R}^r$ are (real) linearly independent, where each $\vec{a}_i$ is defined by

$$\vec{a}_i = \left(\xi^\nu(\psi_i)\right)_{\nu=1}^r \in \mathbb{R}^r \quad \text{for each } i = m+1, \ldots, m+q.$$

Consequently, there exist q-many coordinates such that the restrictions $\vec{a}_i|$ of $\vec{a}_i$ $(m+1 \leqslant i \leqslant m+q)$ to these coordinates are linearly independent. Without loss of generality, let us assume that they are the first q coordinates; thus,

$$\det\left(\xi^\nu(\psi_i)\right)_{1\leqslant\nu\leqslant q,\ m+1\leqslant m+q} \neq 0. \tag{3.63}$$

Therefore there exist real numbers $(\lambda'_1, \ldots, \lambda'_q)$ such that

$$\sum_{v=1}^{q} \lambda'_v \xi^v(\psi_i) = -\mathrm{Re}\,\Phi(\psi_i) \quad \text{for } i = m+1, \ldots, m+q. \tag{3.64}$$

Note that, for $i = 1, 2, \ldots, m$, both sides of (3.64) are equal to zero thanks to (3.56) and (3.61). Therefore

$$\sum_{v=1}^{q} \lambda'_v \xi^v(p) + \mathrm{Re}\,\Phi(p) = 0 \tag{3.65}$$

for each $p \in \{\psi_1, \ldots, \psi_m, \psi_{m+1}, \ldots, \psi_{m+q}\}$. In view of (3.65), it is clear that, to complete the proof, it suffices to show that the first summation in (3.65) can be expressed in the form

$$\sum_{v=1}^{q} \lambda'_v \xi^v(p) = \mathrm{Re} \sum_{j=1}^{s} p(t''_j)\overline{\tau''_j} \quad \text{for each } p \in \mathcal{P} \tag{3.66}$$

for some $s \leqslant q$, $\{t''_j\}_{j=1}^s \subseteq Q_E \cup Q_S$, $\{\tau''_j\}_{j=1}^s \subset \mathbb{C} \setminus \{0\}$ such that

$$\tau''_j \in -N_{\Omega_{t''_j}}(p^*(t''_j)) \quad \text{for each } j = 1, 2, \ldots, s. \tag{3.67}$$

To do this, we consider, in light of (3.62), v with $1 \leqslant v \leqslant q$ for each of the following cases.

(a) $\xi^v = \xi_t$ for some $t \in Q_E$. Then $\tau''_t := \lambda'_v \tau_t \in -N_{\Omega_t}(p^*(t))$ by (3.58), and by (3.59),

$$(\lambda'_v \xi^v)(p) = \lambda'_v \mathrm{Re}\,\overline{\tau_t}\, p(t) = \mathrm{Re}\, p(t)\overline{\tau''_t} \quad \text{for each } p \in \mathcal{P}.$$

(b) $\xi^v = \xi_t$ for some $t \in Q_S$ but $\xi'_t \notin \{\xi^1, \xi^2, \ldots, \xi^r\}$. Then $\tau''_t := \lambda'_v \in -N_{\Omega_t}(p^*(t))$ as $\Omega_t = \{p^*(t)\}$, and

$$(\lambda'_v \xi^v)(p) = \lambda'_v \mathrm{Re}\, p(t) = \mathrm{Re}\, p(t)\overline{\tau''_t} \quad \text{for each } p \in \mathcal{P}.$$

(c) $\xi^v = \xi'_t$ for some $t \in Q_S$ but $\xi_t \notin \{\xi^1, \xi^2, \ldots, \xi^r\}$. Then $\tau''_t := \overline{\mathrm{i}\lambda'_v} \in -N_{\Omega_t}(p^*(t))$ and

$$(\lambda'_v \xi^v)(p) = \lambda'_v \mathrm{Re}\,\mathrm{i}\, p(t) = \mathrm{Re}\, p(t)\overline{\tau''_t} \quad \text{for each } p \in \mathcal{P}.$$

(d) $\xi^v = \xi_t$ for some $t \in Q_S$ which satisfies an additional property that $\xi'_t \in \{\xi^1, \xi^2, \ldots, \xi^r\}$. Assume that $\xi'_t = \xi^{v'}$. Then $\tau''_t := \lambda'_v + \overline{\mathrm{i}\lambda'_{v'}} \in -N_{\Omega_t}(p^*(t))$ as $\Omega_t = \{p^*(t)\}$, and

$$\lambda'_v \xi^v(p) + \lambda'_{v'} \xi^{v'}(p) = \lambda'_v \mathrm{Re}\, p(t) + \lambda'_{v'} \mathrm{Re}\,\bar{\mathrm{i}}\, p(t) = \mathrm{Re}\, p(t)\overline{\tau''_t} \quad \text{for each } p \in \mathcal{P}.$$

Combining (a–d) and deleting these terms with the corresponding $\tau''_t = 0$, (3.66) is seen to hold. □

In the following Theorems 3.1–3.5, we consider relations of the following statements for a fixed pair of functions $f \in C(Q)$ and $p^* \in \mathcal{P}_\Omega$. Recall that $\sigma := f - p^*$.

(i) p^* is a best-restricted range approximation to f from $\mathcal{P}$ with respect to $\{\Omega_t\}$.
(ii) For each $p \in \mathcal{P}_R$, there exist $t \in M(\sigma),\ t' \in B^{rb}(p^*)$ such that

$$\max\left\{\mathrm{Re}\,(p(t)\overline{\sigma(t)}),\ \max_{\tau\in\tau_r^+(t')} \mathrm{Re}\,(p(t')\overline{\tau})\right\} \geqslant 0. \tag{3.68}$$

(iii) For each $p \in \mathcal{P}_R$, there exist $t \in M(\sigma),\ t' \in B^{rb}(p^*)$ and $\tau \in \tau_r(t')$ such that

$$\max\{\mathrm{Re}\,(p(t)\overline{\sigma(t)}), \mathrm{Re}\,(p(t')\overline{\tau})\} \geqslant 0. \tag{3.69}$$

(iv) The origin of $\mathbb{R}^m$ belongs to the convex hull of the $\mathcal{W}_r^+$.
(v) The origin of $\mathbb{R}^m$ belongs to the convex hull of the $\mathcal{W}_r$.
(vi) The origin of $\mathbb{C}^n$ belongs to the convex hull of the $\mathcal{W}$.
(vii) There exist

$$\{t_i\}_{i=1}^k \subseteq M(\sigma), \quad \{\lambda_i\}_{i=1}^k \subset (0, +\infty)$$

and

$$\{t'_j\}_{j=1}^l \subseteq B^{rb}(p^*), \quad \{\tau'_j\}_{j=1}^l \subset \mathbb{C} \setminus \{0\}$$

with $1 + l \leqslant k + l \leqslant m + 1$ such that $\tau'_j \in -N_{\Omega_{t'_j}}(p^*(t'_j))$ for each $j = 1, \ldots, l$, and

$$\mathrm{Re} \sum_{i=1}^k \lambda_i p(t_i)\overline{\sigma(t_i)} + \mathrm{Re} \sum_{j=1}^l p(t'_j)\overline{\tau'_j} = 0 \quad \text{for each } p \in \mathcal{P}_R. \tag{3.70}$$

(viii) There exist

$$\{t_i\}_{i=1}^k \subseteq M(\sigma), \quad \{\lambda_i\}_{i=1}^k \subset (0, +\infty) \tag{3.71}$$

and

$$\{t'_j\}_{j=1}^{l'} \subseteq B(p^*), \quad \{\tau'_j\}_{j=1}^{l'} \subset \mathbb{C} \setminus \{0\} \tag{3.72}$$

with $1+l' \leqslant k+l' \leqslant 2n+1$ such that $\tau'_j \in -N_{\Omega_{t'_j}}(p^*(t'_j))$ for each $j = 1, \ldots, l'$, and

$$\sum_{i=1}^k \lambda_i p(t_i)\overline{\sigma(t_i)} + \sum_{j=1}^{l'} p(t'_j)\overline{\tau'_j} = 0 \quad \text{for each } p \in \mathcal{P}. \tag{3.73}$$

(ix) $J(\sigma)|_{\mathcal{P}} \bigcap \left(\sum_{t\in Q} N_{C_t}(p^*)|_{\mathcal{P}}\right) \neq \emptyset$.

Theorem 3.1. *The following implications hold.*

$$\begin{array}{ccccccccccc}
(\text{vii}) & \Longleftrightarrow & (\text{viii}) & \Longleftrightarrow & (\text{ix}) & \Longrightarrow & (\text{iv}) & \Longrightarrow & (\text{ii}) & \Longleftrightarrow & (\text{iii}) \\
 & & & & \Downarrow & & \Updownarrow & & & & \\
 & & & & (\text{i}) & & (\text{v}) & \Longrightarrow & (\text{vi}) & &
\end{array}$$

Proof. By (3.29), it is easy to verify that (iv) $\Longleftrightarrow$ (v). Also, by (3.17) and (3.18), we have (ii) $\Longleftrightarrow$ (iii). Applying Lemma 3.6 to the functional Φ on $\mathcal{P}$ defined by

$$\Phi(p) = \sum_{i=1}^{k} \lambda_i p(t_i)\overline{\sigma(t_i)} + \mathrm{Re} \sum_{j=1}^{l} p(t'_j)\overline{\tau'_j} \quad \text{for each} \quad p \in \mathcal{P},$$

we have that (vii) $\Longrightarrow$ (viii) with $l' = l + s$, where s is as in Lemma 3.6. To show (viii) $\Longrightarrow$ (vii) $\Longrightarrow$ (v), we suppose that (viii) holds. Thus we assume that (3.73) holds with appropriate $k,\ l',\ \{t_i\},\ \{\lambda_i\},\ \{t'_j\}$ and $\{\tau'_j\}$ as stated in (viii). Without loss of generality, assume that $\{t'_1, \dots, t'_l\} \subseteq B^{rb}(p^*),\ \{t'_{l+1}, \dots, t'_{l'}\} \subseteq B(p^*) \setminus B^{rb}(p^*)$. Note that if $l + 1 \leqslant j \leqslant l'$, then $t_{j'} \in Q_S \cup Q_E$, and $\Omega_{t'_j}$ is either a singleton or a line-segment containing $p^*(t_j)$ as an internal point (seeing (3.9)). Hence

$$\mathrm{Re}\, \overline{\tau'_j}\alpha = 0 \quad \text{for each } \alpha \in \mathrm{span}_R(\Omega_{t'_j} - p^*(t'_j)), \qquad j = l+1, \dots, l'. \tag{3.74}$$

This implies that, for each $p \in \mathcal{P}_R$, $\mathrm{Re}\, \overline{\tau'_j} p(t'_j) = 0$ if $l + 1 \leqslant j \leqslant l'$ because $p(t'_j) \in \mathrm{span}_R(\Omega_{t'_j} - p^*(t'_j))$ by (3.10). Consequently, (3.73) implies that

$$\mathrm{Re} \sum_{i=1}^{k} \lambda_i p(t_i)\overline{\sigma(t_i)} + \mathrm{Re} \sum_{j=1}^{l} p(t'_j)\overline{\tau'_j} = 0 \quad \text{for each } p \in \mathcal{P}_R. \tag{3.75}$$

Replacing $\lambda_i,\ t'_j$ by their appropriate positive multipliers if necessary, we can assume that $k + l \leqslant m + 1$. To see this, we note first that if $\frac{\tau'_j}{|\tau'_j|} \in \tau(t'_j) \setminus \tau_r(t'_j)$, then (3.16), (3.13) and (3.10) imply that $\mathrm{Re}\, p(t'_j)\overline{\tau'_j} = 0$ for each $p \in \mathcal{P}_R$ and thus the corresponding term in (3.75) can be deleted. Henceforth, we suppose therefore that each $\frac{\tau'_j}{|\tau'_j|} \in \tau_r(t_j)$ in (3.75). Noting that $k \geqslant 1$ from the assumption and recalling definitions (3.20) and (3.25), it follows from (3.75) (applied to $\psi_1, \dots, \psi_m$ in place of p) that

$$-\mathbf{b}_r(t_1) \in \mathrm{cone}\,\{\mathbf{b}_r(t_2), \dots, \mathbf{b}_r(t_k), \mathbf{c}_r(t'_1), \dots, \mathbf{c}_r(t'_l)\} \subseteq \mathbb{R}^m.$$

Consequently, by [19, Corollary 17.1.2], $-\mathbf{b}_r(t_1)$ can be expressed as a linear combination of at most m elements from $\{\mathbf{b}_r(t_2), \dots, \mathbf{b}_r(t_k), \mathbf{c}_r(t'_1), \dots, \mathbf{c}_r(t'_l)\}$ with positive coefficients. Thus, appropriately redefining λ_i and t'_j if necessary, we can assume that, $k + l \leqslant m + 1$, (3.75) holds for each $p \in \{\phi_1, \dots, \phi_m\}$ and hence for all $p \in \mathcal{P}_R$. Therefore (viii) $\Longrightarrow$ (vii) and hence (viii) $\Longleftrightarrow$ (vii).

For (vii) $\Longrightarrow$ (v) & (i), suppose that (3.70) holds with appropriate $\{t_i\},\ \{\lambda_i\},\ \{t'_j\}$ and $\{\tau'_j\}$ given in (vii). By an earlier argument, we may assume that $\{t'_1, \dots, t'_r\} \subseteq Q_N,\ \{t'_{r+1}, \dots, t'_l\} \subseteq Q_E$ and $\frac{\tau'_j}{|\tau'_j|} \in \tau_r(t_j)$ for each $r + 1 \leqslant j \leqslant l$. Thus, (3.70) means that the origin of $\mathbb{R}^m$ belongs to the convex hull of the $\mathcal{W}_r$. Consequently, (v) holds. We go on to show that **(i)** holds. To this end, let $p \in \mathcal{P}_\Omega$. Then $p^* - p \in \mathcal{P}_R$ and

$$\mathrm{Re}\sum_{i=1}^{k} \lambda_i (p^* - p)(t_i)\overline{\sigma(t_i)} + \mathrm{Re}\sum_{j=1}^{l} (p^* - p)(t'_j)\overline{\tau'_j} = 0. \tag{3.76}$$

Since $k \geqslant 1$ and each $\lambda_i > 0$, we assume without loss of generality that $\sum_{i=1}^{k} \lambda_i = 1$. Thus, $\|f - p\| \geqslant \sum_{i=1}^{k} \lambda_i |(f - p)(t_i)|^2$. Since $p \in P_\Omega$ and $\tau'_j \in -N_{\Omega_{t'_j}}(p^*(t'_j))$, one has that

$$\mathrm{Re}\,(p^* - p)(t'_j)\overline{\tau'_j} \leqslant 0, \quad j = 1, 2, \ldots, l. \tag{3.77}$$

Hence

$$\begin{aligned}
\|f - p\|^2 &\geqslant \sum_{i=1}^{k} \lambda_i |(f - p)(t_i)|^2 + 2\mathrm{Re}\sum_{j=1}^{l} (p^* - p)(t'_j)\overline{\tau'_j} \\
&= \sum_{i=1}^{k} \lambda_i |(f - p^*)(t_i)|^2 + \sum_{i=1}^{k} \lambda_i |(p^* - p)(t_i)|^2 \\
&\quad +2\mathrm{Re}\sum_{i=1}^{k} \lambda_i (p^* - p)(t_i)\overline{(f - p^*)(t_i)} + 2\mathrm{Re}\sum_{j=1}^{l} (p^* - p)(t'_j)\overline{\tau'_j} \\
&= \sum_{i=1}^{k} \lambda_i |(f - p^*)(t_i)|^2 + \sum_{i=1}^{k} \lambda_i |(p^* - p)(t_i)|^2 \\
&\geqslant \|f - p^*\|^2,
\end{aligned}$$

where the second equality holds because of (3.76) while the last inequality holds because $\{t_i\} \subseteq M(\sigma)$. This means that p^* is a best approximation to f from P_Ω and hence (i) holds.

For (v) $\Longrightarrow$ (vi) & (ii), suppose that there exist nonnegative integers k, l with $k + l \geqslant 1$ such that

$$0 \in \mathrm{conv}\,\{\mathbf{b}_r(t_1), \mathbf{b}_r(t_2), \ldots, \mathbf{b}_r(t_k), \mathbf{c}_r(t'_1), \ldots, \mathbf{c}_r(t'_l)\} \subseteq \mathbb{R}^m \tag{3.78}$$

for some $\{t_i\}_{i=1}^{k} \subseteq M(\sigma)$ and $\{t'_j\}_{j=1}^{l} \subseteq B^{rb}(p^*)$. By the Caratheodory Theorem (cf. [2] or [21, p. 73]), we assume without loss of generality that $k + l \leqslant m + 1$. Moreover, by (3.17), (3.20) and (3.25), there exist $\{\lambda_i\} \subset (0, +\infty)$ and $\{\tau'_j\} \subset \mathbb{C} \setminus \{0\}$ with $\tau'_j \in -N_{\Omega_{t'_j}}(p^*(t'_j)) \setminus \{0\}$ for each j such that (3.70) holds for each $p \in \{\psi_1, \ldots, \psi_m\}$ and hence for each $p \in \mathcal{P}_R$. (Note: Since k may be zero, we cannot conclude that (vii) holds.) Now by applying Lemma 3.6 to the functional Φ: $\mathcal{P}_R \to \mathbb{C}$ defined by

$$\Phi(p) = \sum_{i=1}^{k} \lambda_i\, p(t_i)\overline{\sigma(t_i)} + \sum_{j=1}^{l} p(t'_j)\overline{\tau'_j} \quad \text{for each} \quad p \in \mathcal{P}$$

we conclude that (3.57) holds with appropriate $\{t''_j\}$, $\{\tau''_j\}$ stated in Lemma 3.6. By the Caratheodory Theorem, we assume that $k + l + s \leqslant 2n + 1$. Thus we see that (vi) holds (dividing both sides of (3.57) by a positive constant if necessary). Note, in passing, again that (viii) would hold provided that $k \neq 0$. Moreover, (ii) must also hold because otherwise there exists an element $p_0 \in \mathcal{P}_R$ such that

$$\max\{\mathrm{Re}\,(p_0(t_i)\overline{\sigma(t_i)}), \mathrm{Re}\,(p_0(t'_j)\overline{\tau'_j})\} < 0 \quad \text{for each } i = 1, \ldots, k \text{ and } j = 1, \ldots, l. \tag{3.79}$$

This contradicts (3.70) as the number on the left-hand side of (3.70) with $p = p_0$ is negative by (3.79). Hence, the proof of (v) $\Longrightarrow$ (vi) & (ii) is complete.

Finally we show that (viii) $\Longleftrightarrow$ (ix). Suppose first that (ix) holds. Then, there exist $v^* \in J(\sigma)$ and $w_j^* \in N_{C_{t'_j}}(p^*)$, $j = 1, 2, \ldots, s$ with $p^* \in \mathrm{bd}\, C_{t'_j}$ (namely $t'_j \in B(p^*)$ such that

$$\langle v^*, p\rangle = \sum_{j=1}^{s} \langle w_j^*, p\rangle \quad \text{for all } p \in \mathcal{P}. \tag{3.80}$$

Set $u^* = v^*/\|v^*\|$. Applying [22, Lemma 1.3, p. 169] to the real linear span of $\mathcal{P} \cup \{f\}$, there exist a positive integer r (with $1 \leqslant r \leqslant 2(n+1)$), r extreme points $u_1^*, \ldots, u_r^*$ of the unit ball Σ^* of $C(Q)^*$ and positive constants β_i, $i = 1, 2, \ldots, r$, with $\sum_{i=1}^{r} \beta_i = 1$ such that

$$\langle u^*, p\rangle = \sum_{i=1}^{r} \beta_i \langle u_i^*, p\rangle \quad \text{for all } p \in \mathrm{span}\,(\mathcal{P} \cup \{f\}). \tag{3.81}$$

By a well-known representation of the extreme points of Σ^* (cf. [22, p. 69]), there exist some $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1$ and $t_i \in Q$ such that

$$u_i^* = \alpha_i e_{t_i}, \quad i = 1, 2, \ldots, r.$$

By the definition of u^*, $\|u^*\| = 1$ and $\langle u^*, \sigma\rangle = \|\sigma\|$; hence, by (3.81), $t_i \in M(\sigma)$ and $\alpha_i = \overline{\sigma(t_i)}/\|\sigma\|$. Furthermore, by (3.41), for each j, there exists $\tau'_j \in -N_{\Omega_{t'_j}}(p^*(t'_j))$ such that $-w_j^* = \overline{\tau'}_j e_{t'_j}$. Therefore, (3.80) becomes

$$\sum_{i=1}^{r} \beta'_i p(t_i)\overline{\sigma(t_i)} + \sum_{j=1}^{s} p(t'_j)\overline{\tau'_j} = 0 \quad \text{for all } p \in \mathcal{P}, \tag{3.82}$$

where $\beta'_i = \|v^*\|\beta_i/\|\sigma\|$ for each $i = 1, \ldots, r$. Set

$$\mathbf{c}_j = (\phi_1(t), \ldots, \phi_n(t))\overline{\tau'_j} \quad \text{for each } j = 1, \ldots, s.$$

Then (3.82) implies that

$$-\beta'_1\mathbf{b}(t_1) \in \mathrm{cone}\{\beta'_2\mathbf{b}(t_2), \ldots, \beta'_r\mathbf{b}(t_r), \mathbf{c}_1, \ldots, \mathbf{c}_s\}.$$

Since $\dim_R \mathcal{P} = 2n$, by [19, Corollary 17.1.2], $-\beta'_1\mathbf{b}(t_1)$ can be expressed as a linear combination of at most $2n$ elements from $\{\beta'_2\mathbf{b}(t_2), \ldots, \beta'_r\mathbf{b}(t_r), \mathbf{c}_1, \ldots, \mathbf{c}_s\}$ with positive coefficients. Hence, replacing β'_i and τ'_j by their appropriate positive multipliers we can assume without loss of generality that r, s in (3.82) satisfy the additional property that

$1 + s \leqslant r + s \leqslant 2n + 1$. Thus (viii) holds with $(k, l') = (r, s)$. Conversely, suppose that (viii) holds. Hence we have (3.73) with appropriate $\{t_i\}_{i=1}^k$, $\{\lambda_i\}_{i=1}^k$ and $\{t'_j\}_{j=1}^{l'}$, $\{\tau'_j\}_{j=1}^{l'}$ as stated in (viii). We can of course assume that $\sum_{i=1}^k \lambda_i = 1$, and rewrite (3.73) as

$$\sum_{i=1}^{k} \lambda_i \overline{\sigma(t_i)} e_{t_i} = -\sum_{j=1}^{l'} \overline{\tau'_j} e_{t'_j} \ (\in \mathcal{P}^*). \tag{3.83}$$

By Lemma 3.2, $\overline{\tau'_j} e_{t'_j} \in N_{C_{t'_j}}(p^*)$ for each $j = 1, 2, \ldots, l'$. On the other hand, since $t_i \in M(\sigma)$, we have that $\langle \overline{\sigma(t_i)} e_{t_i}, \sigma \rangle = \|\sigma\|^2$ for each $i = 1, 2, \ldots, k$. Therefore the functional expressed by either side of (3.83) belongs to the intersection in (ix). □

Theorem 3.2. *It holds that* (v) $\Longleftrightarrow$ (vii) *if* IC *is assumed, and that* (vi) $\Longleftrightarrow$ (viii) *if* IC_0 *is assumed.*

Proof. Suppose that (v) holds and we proceed as in the proof for (v) $\Longrightarrow$ (vi) & (ii) of Theorem 3.1. If IC is assumed in addition, $0 \notin \text{conv}\,\mathcal{U}_r$ by Lemma 3.1. Hence k in (3.78) must be nonzero and so (vii) holds. Similarly, suppose that (vi) holds (thus, with the exception that k is possibly zero, (3.73) holds). Suppose further that IC_0 is assumed (instead of IC). Then $0 \notin \text{conv}\,\mathcal{U}$ by Lemma 3.1. Hence k in (3.73) must be nonzero. Therefore (viii) holds. □

Theorem 3.3. *If the system* $\{\mathcal{P}, C_t : t \in Q\}$ *has the strong CHIP at* p^*, *then* (i) $\Longleftrightarrow$ (vii).

Proof. Note that $\mathcal{P}_\Omega = \mathcal{P} \cap (\cap_{t \in Q} C_t)$. By the implication (i) $\Longleftrightarrow$ (iv) in Theorem 2.3 and the fact that $\mathcal{P}$ is a vector subspace containing p^* (so $N_{\mathcal{P}}(p^*)|_{\mathcal{P}} = 0$), we now have that (i) $\Longleftrightarrow$ (ix) thanks to the strong CHIP assumption. Since (ix) $\Longleftrightarrow$ (vii) by Theorem 3.1, (i) $\Longleftrightarrow$ (vii) holds. □

Theorem 3.4. *If both* LKC *and* IC *are assumed, then the statements in the list* (i)–(ix) *except* (vi) *are equivalent to each other.*

Proof. Suppose that both LKC and IC hold. We will show that the CCS-system $\{\mathcal{P}, C_t : t \in Q\}$ has the strong CHIP at p^*. For this purpose, note that, by Lemma 3.3 and Remark 2.1, the condition LKC implies that the set-valued function $t \mapsto C_t$ is lower semicontinuous on Q while, by Lemma 3.1, the condition IC implies that the system $\{\mathcal{P}, C_t : t \in Q\}$ satisfies the weak–strong interior-point condition with (Q_E, Q_S). By Theorem 2.1, the system $\{\mathcal{P}, C_t : t \in Q\}$ has the strong CHIP at p^*. By Theorems 3.3, 3.1 and 3.2, it remains to show that (ii) $\Longleftrightarrow$ (v). Suppose on the contrary that (ii) holds but (v) is false. Then, by Lemma 3.5, $0 \notin \text{conv}\,\overline{\mathcal{W}_r^+}$ ($\subseteq \text{conv}\,\mathcal{W}_r$). By the Linear Inequality Theorem (see [2]), there exists $z^0 = (\gamma_1^0, \ldots, \gamma_m^0) \in \mathbb{R}^m$ such that

$$\langle u, z^0 \rangle < 0 \quad \text{for all } u \in \overline{\mathcal{W}_r^+}. \tag{3.84}$$

Then $\max_{u\in\overline{\mathcal{W}_r^+}}\langle u, z^0\rangle < 0$ because $\overline{\mathcal{W}_r^+}$ is compact (noting that $\mathcal{W}_r^+$ is bounded). Let $p_0 = \sum_{i=1}^m \gamma_i^0\psi_i$. Then $p_0 \in \mathcal{P}_R$. By (3.25) and (3.21), for any $t \in M(\sigma)$, $t' \in B^{rb}(p^*)$ and $\tau \in \tau_r^+(t')$, one has

$$\mathrm{Re}\,(p_0(t)\overline{\sigma(t)}) = \langle \mathbf{b}_r(t), z^0\rangle, \qquad \mathrm{Re}\,(p_0(t')\overline{\tau}) = \langle u_\tau, z^0\rangle,$$

where $u_\tau \in \mathbf{c}_r^+(t')$ is defined by $u_\tau := (\mathrm{Re}\,\psi_1(t')\overline{\tau}, \ldots, \mathrm{Re}\,\psi_m(t')\overline{\tau})$. Since $\{\mathbf{b}_r(t)\} \cup \mathbf{c}_r^+(t') \subseteq \mathcal{W}_r^+$, we have that

$$\max\left\{\mathrm{Re}\,(p_0(t)\overline{\sigma(t)}),\ \max_{\tau\in\tau_r^+(t')}\mathrm{Re}\,(p_0(t')\overline{\tau})\right\}$$
$$= \max\left\{\langle \mathbf{b}_r(t), z^0\rangle,\ \max_{\tau\in\tau_r^+(t')}\langle u_\tau, z_0\rangle\right\} \leqslant \max_{u\in\overline{\mathcal{W}_r^+}}\langle u, z^0\rangle < 0,$$

which contradicts (ii). □

Theorem 3.5. *If both* KC *and* IC_0 *are assumed, then the statements* (i)–(ix) *are mutually equivalent.*

Proof. Suppose that both KC and IC_0 hold. Then $\mathcal{W}$ is compact in $\mathbb{C}^n$ by Lemma 3.4. Using this, and similar arguments as in the proof of (ii)⟹(v) in Theorem 3.4 give that (ii) ⟺ (vi) (use W, $\mathbb{C}^n$ and $\mathrm{Re}\,\langle u, z\rangle$ to replace $\overline{W_r^+}$, $\mathbb{R}^m$ and $\langle u, z\rangle$). By Theorem 3.2, (vi) ⟺ (viii). Thus, by Theorem 3.1, it remains to show that (i) ⟺ (vii). In view of Theorem 3.3, it suffices to show that the CCS-system $\{\mathcal{P}, C_t : t \in Q\}$ has the strong CHIP at p^*. But this follows easily from Theorem 2.2 which is applicable to this system by Lemma 3.1(i) and Lemma 3.3 (thanks to the assumptions). □

Acknowledgments

Chong Li was supported in part by the National Natural Science Foundation of China (Grant 10271025) and Program for New Century Excellent Talents in University. K.F. Ng was supported by a direct grant (CUHK) and an Earmarked Grant from the Research Grant Council of Hong Kong.

References

[1] D. Braess, Nonlinear Approximation Theory, Springer, New York, 1986.
[2] E.W. Cheney, Introduction to Approximation Theory, McGraw-Hill, New York, 1966.
[3] C. Chui, F. Deutsch, J. Ward, Constrained best approximation in Hilbert space, Constr. Approx. 6 (1990) 35–64.
[4] C. Chui, F. Deutsch, J. Ward, Constrained best approximation in Hilbert space II, J. Approx. Theory 71 (1992) 231–238.
[5] F. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[6] F. Deutsch, Some applications of functional analysis to approximation theory, Doctoral dissertation, Brown University, 1965.

[7] F. Deutsch, W. Li, J. Ward, A dual approach to constrained interpolation from a convex subset of Hilbert space, J. Approx. Theory 90 (1997) 385–414.
[8] F. Deutsch, W. Li, J. Ward, Best approximation from the intersection of a closed convex set and a polyhedron in Hilbert space, weak Slater conditions, and the strong conical hull intersection property, SIAM J. Optim. 10 (1999) 252–268.
[9] F. Deutsch, V. Ubhaya, J. Ward, Y. Xu, Constrained best approximation in Hilbert space III: application to n-convex functions, Constr. Approx. 12 (1996) 361–384.
[10] J. Hiriart-Urruty, C. Lemaréchal, Convex Analysis and Minimization Algorithms I, Grundlehren der Mathematschen Wissenschaften, vol. 305, Springer, New York, 1993.
[11] C. Li, On best uniform restricted range approximation in complex-valued continuous function spaces, J. Approx. Theory 120 (2003) 71–84.
[12] C. Li, X.Q. Jin, Nonlinearly constrained best approximation in Hilbert spaces, the strong conical hull intersection property and the basic constraints qualification condition, SIAM J. Optim. 13 (2002) 228–239.
[13] C. Li, K.F. Ng, On best approximation by nonconvex sets and perturbation of nonconvex inquality systems in Hilbert spaces, SIAM J. Optim. 13 (2002) 726–744.
[14] C. Li, K.F. Ng, Constraint qualification, the strong CHIP and best approximation with convex constraints in Banach spaces, SIAM J. Optim. 14 (2003) 584–607.
[15] C. Li, K.F. Ng, On constraint qualification for infinite system of convex inequalities in a Banach space, SIAM J. Optim. 15 (2005) 488–512.
[16] C. Li, K.F. Ng, Strong CHIP for infinite system of closed convex sets in normed linear spaces, SIAM J. Optim., to appear.
[17] C. Micchelli, P. Smith, J. Swetits, J. Ward, Constrained L_p-approximation, Constr. Approx. 1 (1985) 93–102.
[18] C. Micchelli, F. Utreras, Smoothing and interpolation in a convex subset of a Hilbert space, SIAM J. Sci. Statist. Comput. 9 (1988) 728–746.
[19] R. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1970.
[20] G.Sh. Rubenstein, On an extremal problem in a normed linear space, Sibirskii Mat. Zh. 6 (1965) 711–714 (in Russian).
[21] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1981.
[22] I. Singer, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer, Berlin, Heidelberg, New York, 1970.
[23] I. Singer, The Theory of Best Approximation and Functional Analysis, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 13, SIAM, Philadelphia, PA, 1974.
[24] G.S. Smirnov, R.G. Smirnov, Best uniform restricted range approximation of complex-valued functions, C. R. Math. Acad. Sci. Canada 19 (2) (1997) 58–63.
[25] G.S. Smirnov, R.G. Smirnov, Best uniform approximation of complex-valued functions by generalized polynomials having restricted range, J. Approx. Theory 100 (1999) 284–303.
[26] G.S. Smirnov, R.G. Smirnov, Kolmogorov-type theory of best restricted approximation, Eastern J. Approx. 6 (3) (2000) 309–329.
[27] G.S. Smirnov, R.G. Smirnov, Best restricted approximation of complex-valued functions II, C. R. Acad. Sci. Ser. I, Math. 330 (12) (2000) 1059–1064.
[28] G.S. Smirnov, R.G. Smirnov, Theory of best restricted ranges approximation revisited: a characterization theorem, Dziadyk Conference Proceedings, Proceedings of the Institute of Mathematics of NAS of Ukraine, vol. 31, 2000, pp. 436–445.